JOURNAL OF THE CHUNGCHEONG MATHEMATICAL SOCIETY Volume **31**, No. 3, August 2018 http://dx.doi.org/10.14403/jcms.2018.31.1.281

ON THE MINUS PARTS OF CLASSICAL POINCARÉ SERIES

SoYoung Choi*

ABSTRACT. Let $S_k(N)$ be the space of cusp forms of weight k for $\Gamma_0(N)$. We show that $S_k(N)$ is the direct sum of subspaces $S_k^+(N)$ and $S_k^-(N)$. Where $S_k^+(N)$ is the vector space of cusp forms of weight k for the group $\Gamma_0^+(N)$ generated by $\Gamma_0(N)$ and W_N and $S_k^-(N)$ is the subspace consisting of elements f in $S_k(N)$ satisfying $f|_k W_N = -f$. We find generators spanning the space $S_k^-(N)$ from Poincaré series and give all linear relations among such generators.

1. Introduction and statement of results

Let N be a positive integer and k be a positive even integer greater that 2. Let $S_k(N)$ be the space of cusp forms of weight k for $\Gamma_0(N)$ and $S_k^+(N)$ be the subspace of $S_k(N)$ consisting of cusp forms f with $f|_k W_N = f$, i.e.,

$$S_k^+(N) := \{ f \in S_k(N) | f|_k W_N = f \},\$$

where $W_N = \begin{pmatrix} 0 & -1/\sqrt{N} \\ \sqrt{N} & 0 \end{pmatrix}$ is the Fricke involution. We define the subspace of $S_k(N)$ as:

$$S_k^-(N) := \{ f \in S_k(N) | f|_k W_N = -f \}.$$

For each $f \in S_k(N)$, we see $f = \frac{f+f|_k W_N}{2} + \frac{f-f|_k W_N}{2}$. We notice that $\frac{f+f|_k W_N}{2} \in S_k^+(N)$ and $\frac{f-f|_k W_N}{2} \in S_k^-(N)$. We call the cusp form $\frac{f+f|_k W_N}{2}$ the plus part of f and the cusp form $\frac{f-f|_k W_N}{2}$ the minus part of f. We can show that $S_k(N)$ is the direct sum of subspaces $S_k^+(N)$ and $S_k^-(N)$, that is,

$$S_k(N) = S_k^+(N) \bigoplus S_k^-(N).$$

Received March 19, 2018; Accepted May 30, 2018.

²⁰¹⁰ Mathematics Subject Classification: Primary 11E12, 11F11.

Key words and phrases: weakly holomorphic modular forms, Poincaré series.

SoYoung Choi

Here, in fact $S_k^+(N)$ is the vector space of cusp forms of weight k for the group $\Gamma_0^+(N)$ generated by $\Gamma_0(N)$ and W_N .

For each positive integer m, we define the classical cuspidal Poincaré series P(m, k, N; z) as follows:

$$P(m,k,N;z) := \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma_{0}(N)} \frac{e^{2\pi i m \gamma z}}{(cz+d)^{k}},$$

where $z \in \mathbb{H}$ and \mathbb{H} is the complex upper half plane, and $\Gamma_{\infty} = \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z}\}$. It is well known that the Poincaré series are cusp forms in $S_k(N)$ and the set $\{P(m, k, N; z) \mid m \geq 1\}$ spans the cusp form space $S_k(N)$. In [6], Iwanic raise an open problem that ask to find all linear relations among the Poincaré series. In [7] it is shown that linear relations among the Poincaré series P(m, k, N; z) are given by certain weakly holomorphic modular forms. The Poincaré series P(m, k, N; z) can be splitted into

$$P(m,k,N;z) = P^{+}(m,k,N;z) + P^{-}(m,k,N;z),$$

where $P^+(m,k,N;z) := \frac{P(m,k,N;z)+P(m,k,N;z)|_k W_N}{2}$ and $P^-(m,k,N;z) := \frac{P(m,k,N;z)-P(m,k,N;z)|_k W_N}{2}$. Indeed, $P^+(m,k,N;z)$ is just the Poincaré series related to the group $\Gamma_0^+(N)$ as follows:

$$P^+(m,k,N;z) := \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma_0^+(N)} \frac{e^{2\pi i m \gamma z}}{(cz+d)^k}.$$

In [4] the author and Kim considered the Poincaré series $P^+(m, k, N; z)$ that is the plus part of P(m, k, N; z) and found all linear relations among $P^+(m, k, N; z)$ by giving certain weakly holomorphic modular forms of weight 2 - k for $\Gamma_0^+(N)$. For given $g \in S_k^-(N)$, we have $g(z) = \sum_m a_m P(m, k, N; z)$ for some $a_m \in \mathbb{C}$. Thus we have

$$g(z) = \sum_{m} a_m P^-(m, k, N; z)$$

which implies that the set $\{P^{-}(m,k,N;z) \mid m \geq 1\}$ spans the space $S_{k}^{-}(N)$. In this paper, following the argument in [4, 7] we find all linear relations among the series $P^{-}(m,k,N;z)$. With the notations above we can state the result.

THEOREM 1.1. Let I be a nonempty finite subset of \mathbb{N} . Then the following are equivalent.

(1) $\sum_{m \in I} a_m P^-(m, k, N; z) \equiv 0$ for some $a_m \in \mathbb{C}$.

282

(2) There exists a weakly holomorphic modular form g of weight 2-k for $\Gamma_0(N)$ with the principal part at ∞

$$\sum_{m \in I} \frac{\overline{a_m}}{m^{k-1}} q^{-m}$$

and zero principal part at all cusps except ∞ .

This paper is organized as follows. In Section 2 we give basic properties of Harmonic weak Maass forms. In Section 3 we give the proof of Theorem 1.1.

2. Harmonic weak Maass forms

A smooth function $f : \mathbb{H} \to \mathbb{C}$ is called a harmonic weak Maass form of weight 2 - k for $\Gamma_0(N)$ if it satisfies:

- (1) For all $\gamma \in \Gamma_0(N)$ we have $f|_{2-k}\gamma = f$.
- (2) $\triangle_{2-k}f = 0$, where \triangle_{2-k} is the weight 2-k hyperbolic Laplace operator.
- (3) There is Fourier polynomial $P_f(z) = \sum_{-\infty n \le 0} c_f^+(n) q^n \in \mathbb{C}[q^{-1}]$ such that $f(z) = P_f(z) + O(e^{-\varepsilon y})$ as $y \to \infty$ for some $\varepsilon > 0$. And analogous conditions are required at all cusps. Here $q = e^{2\pi i z}$ as usual.

We denote $H_{2-k}(N)$ the space of harmonic weak Maass forms of weight k for $\Gamma_0(N)$. Let $H_{2-k}^+(N)$ be the subspace of $H_{2-k}(N)$ consisting of elements G satisfying $G|_{2-k}W_N = G$ and $H_{2-k}^-(N)$ be the subspace of $H_{2-k}(N)$ consisting of elements G satisfying $G|_{2-k}W_N = -G$. Then we can show that $H_{2-k}(N)$ is the direct sum of $H_{2-k}^+(N)$ and $H_{2-k}^-(N)$.

Harmonic weak Maass forms are related to cusp forms in terms of differential operator. We define a differential operator

$$\xi_v := 2iy^v \frac{\overline{\partial}}{\partial \overline{z}}$$

Then we have

(2.1)
$$\xi_{2-k}(G|_{2-k}\gamma) = (\xi_{2-k}G)|_k\gamma \quad \text{for all } \gamma \in SL_2(\mathbb{R}).$$

PROPOSITION 2.1. The differential operator ξ_{2-k} define a surjective map from $H^-_{2-k}(N)$ to $S^-(N)$ and the kernel of ξ_{2-k} is the space of weakly holomorphic modular forms f satisfying $f|_{2-k}W_N = -f$. SoYoung Choi

Proof. Given $g \in S_k^-(N)$, take $G \in H_{2-k}(N)$ such that $\xi_{2-k}G = g$. Then $2g = \xi_{2-k}G - (\xi_{2-k}G)|_k W_N = \xi_{2-k}(G - G|_{2-k}W_N)$ and $G - G|_k W_N \in H_{2-k}^-(N)$. So ξ_{2-k} is surjective. It is well known that ker ξ_{2-k} is the space of weakly holomorphic modular forms of weight 2 - k for $\Gamma_0(N)$. Thus ker $\xi_{2-k} \cap H_{2-k}^-$ is the space of weakly holomorphic modular form G of weight 2 - k for $\Gamma_0(N)$ satisfying $G|_{2-k}W_N = G$ since G satisfies $G|_{2-k}W_N = -G$ for $G \in H_{2-k}^-(N)$.

PROPOSITION 2.2. If $f \in H_{2-k}(N)$ is not weakly holomorphic modular form then the principal part of f is non-constant for at least one cusp.

Proof. It follows form the work of Bruinier and Funke [1]. \Box

For $s \in \mathbb{C}$, let

$$\mathcal{M}_{s,\kappa}(y) := |y|^{-\frac{\kappa}{2}} M_{\frac{\kappa}{2}sgn(y), s-\frac{1}{2}}(|y|),$$

where $M_{\mu,\nu}(z)$ is the usual *M*-Whittaker function. We define

$$Q(-m,k,N;z) := \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma_{0}(N)} (\varphi_{-m}^{*}|_{2-k}\gamma)(z),$$

where $\varphi_{-m}^*(z) = \mathcal{M}_{\frac{k}{2},2-k}(-4\pi my)e^{-2\pi imx}$ and $z = x + iy \in \mathbb{H}$. We let

$$Q^{-}(-m,k,N;z) := Q(-m,k,N;z) - Q(-m,k,N;z)|_{2-k}W_N.$$

PROPOSITION 2.3. With the notations above we have the following:

- (i) $Q^{-}(-m,k,N;z) \in H^{-}_{2-k}(N)$ and its principal part at ∞ equals to $\Gamma(k)q^{-m}$ and its principal part at the cusps other than ∞ are constant.
- (*ii*) $\xi_{2-k}(Q^{-}(-m,k,N;z)) = (4\pi m)^{k-1}(k-1)P^{-}(-m,k,N;z).$

Proof. It follows from (2.1) and the results in [2].

3. Proof of Theorem 1.1

We assume that

$$\sum_{m \in I} a_m P^-(m,k,N;z) \equiv 0$$

Define $f \in H^{-}_{k-1}(N)$ by

$$f = \sum_{m \in I} \frac{a_m}{m^{k-1}} Q^-(-m, k, N; z).$$

284

Then we have from Proposition 2.3 that $\xi_{2-k}f = 0$. So by Proposition 2.2, Proposition 2.1 and Proposition 2.3 we see that $f/\Gamma(k)$ is weakly holomorphic modular form of weight 2 - k for $\Gamma_0(N)$ that we desire.

Conversely we assume that f is such a weakly holomorphic modular form. Define \tilde{f} by

$$\widetilde{f} := -f + \frac{1}{\Gamma(k)} \sum_{m \in I} \frac{a_m}{m^{k-1}} Q^-(-m,k,N;z).$$

Then $\tilde{f} \in H^-_{2-k}(N)$ has constant principal parts at all cusps. It then follows from Proposition 2.2 that $\xi_{2-k}(\tilde{f}) = 0$ and hence by Proposition 2.3 we have the desired relation.

References

- J. H. Bruinier and J. Funke, On two geometric theta lifts, Duke Math. J. 125 (2004), 45-90.
- [2] J. H. Bruinier, K. Ono, and R. C. Rhoades, Differential operators for harmonic weak Maass forms and the vanishing of Hecke eigenvalues, Math. Ann. 342 (2008), no. 3, 673-693.
- [3] B. Cho, S. Choi, and C. H. Kim, Harmonic weak Maass-modular grids in higher level cases, Acta Arith. 160 (2013), no. 2, 129-141.
- [4] S. Choi and C. H. Kim, Valence formulas for certain arithmetic groups and their applications, J. Math. Anal. Appl. 420 (2014), no. 1, 447-463.
- [5] S. Choi, C. H. Kim, and K. S. Lee, Arithmetic Properties for the Minus Space of Weakly Holomorphic Modular Forms, preprint.
- [6] H. Iwaniec, *Topics in classical automorphic forms*, Graduate Studies in Mathematics, 17, American Mathematical Society, Providence, RI, 1997.
- [7] C. R. Rhoades, Linear relations among Poincaré series via harmonic weak Maass forms, Ramanujan J. 29 (2012), 311-320.

*

Department of Mathematics Education and RINS, Gyeongsang National University, Jinju 52828, Republic of Korea *E-mail*: mathsoyoung@gnu.ac.kr